TD Intégration (2) - Corrigé partiel

1) **a)** $\left[\frac{\ln(1+t^2)}{2}\right]_0^1 = \frac{\ln(2)}{2}$ **b)** $\left[-\frac{\cos(t)^{-2}}{-2}\right]_0^{\pi/4} = \frac{(\sqrt{2}/2)^{-2}}{2} - \frac{1^{-2}}{2} = \frac{2}{2} - \frac{1}{2} = \frac{1}{2}$

c) $\left[\frac{(\arctan(x))^2}{2}\right]_0^1 = \left(\frac{\pi}{4}\right)^2 \times \frac{1}{2} = \frac{\pi^2}{32}$ **d)** $\left[\frac{2}{3}\sqrt{t^3+8}\right]_0^2 = \frac{2}{3}\left(\sqrt{16}-\sqrt{8}\right) = \frac{8-4\sqrt{2}}{8}$

e) $\int_0^1 (x-1)\cos(x)\,dx = \left[(x-1)\sin(x)\right]_0^1 - \int_0^1 \sin(x)\,dx$ par IPP

$= 0 - \left[-\cos(x)\right]_0^1 = \cos(1) - \cos(0) = \cos(1) - 1$.

f) $\left[-\frac{1}{5}\left(1+\frac{1}{t}\right)^5\right]_1^2 = -\frac{1}{5}\left(\left(\frac{3}{2}\right)^5 - 2^5\right) = \frac{32}{5} - \frac{3^5}{160} = \frac{32\times 32-243}{160} = \frac{1024-243}{160} = \frac{781}{160}$

g) $u = t+1 \rightsquigarrow du = dt$ $\int_0^1 (t+1)^5(t-2)\,dt = \int_1^2 u^5(u-3)\,du = \int_1^2 u^6\,du - 3\int_1^2 u^5\,du$

$= \left[\frac{u^7}{7}\right]_1^2 - 3\left[\frac{u^6}{6}\right]_1^2 = \frac{2^7-1}{7} - \frac{3(2^6-1)}{6} = \frac{127}{7} - \frac{63}{2} = \frac{254-441}{14} = -\frac{187}{14}$

h) $\begin{cases} u = t^3 \\ du = 3t^2\,dt \end{cases}$ $\int_1^2 \frac{1}{t(t^3+1)}\,dt = \int_1^2 \frac{1}{3t^3(t^3+1)}(3t^2\,dt) = \int_1^8 \frac{1}{3u(u+1)}\,du = \frac{1}{3}\int_1^8\left(\frac{1}{u} - \frac{1}{u+1}\right)du$

$= \frac{1}{3}\left[\ln(u) - \ln(u+1)\right]_1^8 = \frac{1}{3}\left(\ln(8) - \ln(9) - \ln(1) + \ln(2)\right)$

$= \frac{3\ln(2) - 2\ln(3) + \ln(2)}{3} = \frac{4\ln(2) - 2\ln(3)}{3}$.

i) $\begin{cases} u = \sqrt{t^2-1} \\ du = \frac{2t}{2\sqrt{t^2-1}}\,dt \\ t^2 = u^2+1 \end{cases}$ $\int_2^{\sqrt{2}} \frac{1}{t\sqrt{t^2-1}}\,dt = \int_2^{\sqrt{2}} \frac{1}{t^2}\left(\frac{t\,dt}{\sqrt{t^2-1}}\right) = \int_{\sqrt{3}}^1 \frac{1}{u^2+1}\,du = \left[\arctan u\right]_{\sqrt{3}}^1$

$= \frac{\pi}{4} - \frac{\pi}{3} = -\frac{\pi}{12}$.

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2) **a)** $\boxed{x \in \mathbb{R} \mapsto \ln(1+e^x) + C}$ **b)** $x \in [1,+\infty[\longmapsto \boxed{\frac{2}{9}\left(x^3+9x-10\right)^{3/2} + C}$

c) $x \in \mathbb{R} \mapsto \boxed{\sqrt{3+x^2} + C}$ ← calculs directs !

d) $\forall x \in \mathbb{R}$, $\int_*^x (t-1)\cos(t)\,dt = \left[(t-1)\sin(t)\right]_*^x - \int_*^x \sin(t)\,dt = (x-1)\sin(x) + \left[\cos(t)\right]_*^x + C$

$\boxed{= (x-1)\sin(x) + \cos(x) + C}$

e) $\forall x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. $\int_*^x \frac{t}{\cos^2(t)}\,dt = \left[t\tan(t)\right]_*^x - \int_*^x \tan(t)\,dt = x\tan(x) + \left[\ln(\cos t)\right]_*^x + C$

$\boxed{= x\tan(x) + \ln(\cos x) + C}$

f) $\forall x \in \mathbb{R}$ $\int_*^x (t+2)^2 e^{-t}\,dt = \left[-(t+2)^2 e^{-t}\right]_*^x + 2\int_*^x (t+2)e^{-t}\,dt = -(x+2)^2 e^{-x} + 2\left[-(t+2)e^{-t}\right]_*^x + 2\int_*^x e^{-t}\,dt + C$

$= -(x+2)^2 e^{-x} - 2(x+2)e^{-x} - 2e^{-x} + C = -e^{-x}\left((x+2)^2 + x + 4\right) + C$

3 d) $S_n \xrightarrow[n\to+\infty]{} \int_0^1 \frac{\ln(1+t)}{1+t}\,dt = \left[\frac{\ln(1+t)^2}{2}\right]_0^1 = \frac{\ln(2)^2}{2}$

e) $S_n = \exp\left(\frac{1}{2n}\sum_{k=1}^{n}\ln(k+n)\right)$

Or $\frac{1}{2n}\sum_{k=1}^{n}\underbrace{\ln(k+n)}_{\ln(n)+\ln(\frac{k}{n}+1)} = \frac{\ln(n)}{2} + \underbrace{\frac{1}{2n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+1\right)}_{\xrightarrow[n\to+\infty]{}\frac{1}{2}\int_0^1 \ln(t+1)\,dt} \xrightarrow[n\to+\infty]{} +\infty$ par opération

D'où $\boxed{S_n \xrightarrow[n\to+\infty]{} +\infty}$ après composition avec l'exponentielle.

f) $S_n = \exp\left(\frac{1}{n}\ln\left(\frac{(n+1)(n+2)\cdots(n+n)}{n\times n\times\cdots\times n}\right)\right)$

$= \exp\left(\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{n+k}{n}\right)\right) = \exp\left(\underbrace{\frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right)}_{\xrightarrow[n\to+\infty]{}\int_0^1\ln(1+t)\,dt}\right)$

Or $\int_0^1 \ln(1+t)\,dt = \int_1^2 \ln(t)\,dt = [t\ln(t)-t]_1^2 = 2\ln 2 - 2 + 1 = 2\ln(2) - 1$.

D'où $S_n \xrightarrow[n\to+\infty]{} e^{2\ln(2)-1} = \frac{2^2}{e} = \frac{4}{e}$.